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SIMPLE WAVES IN A BAROTROPIC VORTEX FLUID LAYER

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Mathematical models that describe a plane-parallel vortex flow in a barotropic fluid layer with a free boundary are considered in a long-wave approximation. The existence theorem of simple-wave solutions of the equations of motion is proved for a class of flows with a monotonic-in-depth velocity profile. In the general case, the anomalous behavior of the simplest waves is shown to be observed: along with simple waves decreasing the fluid level, simple waves increasing the fluid level that are defined for all $t > 0$ can exist as well. For a polytropic equation of state, a class of exact solutions that is described by incomplete beta-functions is found.

Teshukov [1] obtained hyperbolicity conditions of the equations for vortex flows in the case of a monotone-in-depth velocity profile. Exact simple-wave solutions for an incompressible fluid were constructed by Freeman [2] and Blythe et al. [3]. The general existence theorem of simple waves propagating in a layer of an eddying incompressible fluid was proved by Teshukov [4]. For a polytropic equation of state with a polytropic exponent γ close to 1 ($\rho = Cp^{\gamma}$), Sachdev and Varughese [5] revealed the asymptotic behavior of a simple-wave solution near a free boundary. Teshukov [6] studied discontinuous solutions and constructed the mathematical model of a hydraulic jump.

1. Formulation of the Problem. We consider the initial boundary-value problem

$$\begin{gathered} u_T + uu_X + vu_Y + \rho^{-1}p_X = 0, \quad \rho^{-1}p_Y = -g, \quad \rho_T + u\rho_X + v\rho_Y + \rho(u_X + v_Y) = 0, \\ \rho = \rho(p), \quad \rho'(p) > 0, \quad u(X,0,Y) = u_0(X,Y), \quad h(X,0) = h_0(X), \\ Y = 0: \quad v = 0, \qquad Y = h(X,T): \quad p = p_0 = \text{const}, \quad h_T + uh_X = v, \end{gathered} \tag{1.1}$$

that describes the flow of an ideal barotropic fluid layer with a free boundary $Y = h(X,T)$ over a flat bottom in the gravity field (g is the acceleration of gravity). The mathematical model (1.1) appears if one takes into account that $H_0/L_0 \ll 1$, where H_0 and L_0 are the characteristic vertical and horizontal scales, respectively.

After the Eulerian–Lagrangian independent variables x, t, and λ ($X = x$, $T = t$, and $Y = \Phi(x,t,\lambda)$, [1]) are introduced, problem (1.1) is reduced to the Cauchy problem in a fixed domain:

$$\begin{gathered} u_t + uu_x + \Big(\rho\Big(p_0 + \int\limits_0^1 H(x,t,\nu)\,d\nu\Big)\Big)^{-1} \int\limits_0^1 H_x(x,t,\nu)\,d\nu = 0, \quad H_t + (uH)x = 0, \\ u(x,0,\lambda) = u_0(x,\lambda h_0(x)), \quad H(x,0,\lambda) = H_0(x,\lambda) > 0, \quad 0 \leqslant \lambda \leqslant 1, \end{gathered} \tag{1.2}$$

where $H(x,t,\lambda) = \rho(x,t,\lambda)\,g\Phi_\lambda$ is a new desired function.

If u and H is the solution of (1.2), the initial unknowns v, p, ρ, and h and the replacement function $\Phi = \Phi(x,t,\lambda)$ are found from the following relations:

$$\begin{gathered} p(x,t,\lambda) = p_0 + \int\limits_\lambda^1 H(x,t,\nu)\,d\nu, \quad \rho = \rho(p), \quad \Phi = \int\limits_0^\lambda H(x,t,\nu)(\rho g)^{-1}d\nu, \\ v = \Phi_t + u\Phi_x, \quad h = \Phi(x,t,1). \end{gathered} \tag{1.3}$$

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Note that one solution of system (1.2) is the solution $u = u(\lambda)$ and $H = H(\lambda)$, which describes a shear flow (X, Y, T) in initial variables [$u = u(Y)$, $v = 0$, and $h = h_0 = \text{const}$].

We shall define a simple-wave solution. According to [1], system (1.2) has the characteristics $dx/dt = k_i$ of the discrete spectrum and also the characteristics $dx/dt = u(x, t, \lambda)$ ($\lambda = \text{const}$) of the continuous spectrum. The characteristic roots of the discrete spectrum are determined by the equation

$$\rho\Big(p_0 + \int_0^1 H(x, t, \nu)\, d\nu\Big) = \int_0^1 H(x, t, \nu)(u(x, t, \nu) - k_i)^{-2} d\nu, \tag{1.4}$$

which always has, outside the interval of variation of the function $u(x, t, \lambda)$, only two real roots (k_1 and k_2) such that $k_1 < \min\limits_{0 \leqslant \lambda \leqslant 1} u(x, t, \lambda)$ and $k_2 > \max\limits_{0 \leqslant \lambda \leqslant 1} u(x, t, \lambda)$.

We call the solution $u = u(\eta(x, t), \lambda)$ and $H = H(\eta(x, t), \lambda)$, where $\eta(x, t)$ is the function of variables x and t, a simple-wave solution of Eqs. (1.2). In what follows, as a simple-wave parameter, we choose the function of pressure distribution at the bottom:

$$\int_0^1 H(x, t, \lambda)\, d\lambda = \eta(x, t). \tag{1.5}$$

For system (1.1), by virtue of relations (1.3), the simple-wave solution is

$$p = p_1(\eta(X, T), Y), \quad u = u_1(\eta(X, T), Y), \quad v = h_X v_1(\eta(X, T), Y). \tag{1.6}$$

In the present paper, we consider simple waves that are subject to the condition

$$k = -\eta_t/\eta_x \neq u \tag{1.7}$$

for any λ. For simple waves, from (1.2) we obtain the equations

$$u_\eta = -(\rho(p_0 + \eta)(u - k))^{-1}, \qquad H_\eta = H(\rho(p_0 + \eta))^{-1}(u - k)^{-2}. \tag{1.8}$$

With allowance for (1.5), integration of the second equation in (1.8) over λ from 0 to 1 yields that k should satisfy the characteristic equation (1.4). For definiteness, we analyze simple waves that correspond to the right-hand characteristic root $k = k_2$ (the case $k = k_1$ is considered similarly). It is convenient to derive a differential equation for k by differentiating (1.4) on (1.8):

$$k'(\eta) = \Big(\rho' - \frac{3}{\rho(p_0 + \eta)} \int_0^1 H(u - k)^{-4} d\nu\Big) \Big/ 2 \int_0^1 H(u - k)^{-3} d\nu. \tag{1.9}$$

For system (1.8) and (1.9), it is reasonable to formulate the Cauchy problem with the following data for $\eta = \eta_0$ ($\eta_0 = \text{const}$):

$$H(\eta_0, \lambda) = H_0(\lambda), \qquad u(\eta_0, \lambda) = u_0(\lambda), \qquad k(\eta_0) = k_0. \tag{1.10}$$

Here k_0 is the right-hand root of Eq. (1.4) for $u = u_0(\lambda)$ and $H = H_0(\lambda)$ [$k_0 > \max\limits_{0 \leqslant \lambda \leqslant 1} u_0(\lambda)$].

If u, H, and k is the solution of problem (1.8)–(1.10), relation (1.7) is intended for the determination of $\eta = \eta(x, t)$. It follows from (1.7) that η is constant along the characteristics $dx/dt = k_2$ of system (1.2). Problem (1.8)–(1.10) is the problem of contiguity of the continuous simple-wave solution to a given shear flow on the characteristic that corresponds to $\eta = \eta_0$ (η_0 is the constant depth of the shear flow in variables x and λ).

Let us consider an alternative formulation of the problem of finding a simple wave. We change the variables in Eqs. (1.1). Let us denote $\tau = dp/dt = p_T + up_X + vp_Y$. We regard x, p, and t, where $X = x$ and $T = t$, as independent variables, and u, Y, and τ as new unknown quantities. This can be done owing to the monotone dependence of p on Y:

$$p_Y = -\rho(p)\, g. \tag{1.11}$$

System (1.11) is transformed to

$$u_t + uu_x + \tau u_p + gY_x = 0, \quad Y_p = -(\rho(p)\, g)^{-1}, \quad u_x + \tau_p = 0. \tag{1.12}$$

Note that system (1.12) is close to the system of a long-wave approximation of an incompressible fluid [the "velocity" divergence (u, τ) is equal to zero]. In variables (x, p), the unknown boundary $Y = h(X, T)$ becomes known, and the boundary condition is transformed to

$$p = 0: \quad \tau = 0 \tag{1.13}$$

(without loss of generality, one can consider $p_0 = 0$, because the case $p_0 \neq 0$ is similar). In new variables, the known boundary $Y = 0$ becomes unknown. If $\eta(x, t)$ is the pressure distribution at the bottom $Y = 0$, the relation

$$\eta_t + u\eta_x = \tau \tag{1.14}$$

should be satisfied at the unknown boundary $p = \eta(x, t)$.

If u, τ, and η is the solution of problem (1.12)–(1.14), then h is found from the relation

$$h = \int\limits_0^{\eta(x,t)} (\rho(s)g)^{-1} ds, \tag{1.15}$$

obtained by integration of (1.11). In this case, $p(X, Y, T) = F(g(h - Y))$ can be found by inversion of the function

$$h - Y = \int\limits_0^{p} (\rho(s)\, g)^{-1} ds. \tag{1.16}$$

The vertical velocity is given by the relation $v = Y_t + uY_x + \tau Y_p$. In variables ($x$, p, and t), owing to (1.15) and (1.16), the simple wave is specified by the relations $u = u(\eta(x, t), p)$ and $\tau = \eta_x \tau(\eta(x, t), p)$.

By virtue of Eqs. (1.12), one can introduce an analog of the stream function $\Psi = \Psi(\eta(x, t), p)$ using the relations $u = \Psi_p$ and $\tau = -\eta_x \Psi_\eta$. From the condition at the boundary $p = \eta(x, t)$,

$$k = -\eta_t/\eta_x = -(\Psi_p + \Psi_\eta)(\eta, \eta) = -[\Psi(\eta, \eta)]'_\eta. \tag{1.17}$$

With allowance for (1.17), from the first equation in system (1.12) we obtain

$$(\Psi_p - [\Psi(\eta, \eta)]'_\eta)\, \Psi_{p\eta} - \Psi_p \Psi_{pp} + \rho^{-1}(p) = 0. \tag{1.18}$$

At the boundary $p = 0$, the condition ($\tau = 0$)

$$\Psi_\eta(\eta, 0) = 0 \tag{1.19}$$

should be satisfied.

Thus, the problem of finding a simple wave reduces either to the Cauchy problem (1.8)–(1.10) or to the problem (1.18) and (1.19).

3. Existence of Simple Waves. The problem of the existence of simple waves is treated for the monotone velocity profile $u_{0\lambda} > 0$. It follows from (1.8) that $(u_\lambda H^{-1})_\eta = 0$. The relation

$$u_\lambda H^{-1} = u_{0\lambda} H_0^{-1} \tag{2.1}$$

then is the integral of system (1.8) and (1.9). By virtue of (2.1), $u_\lambda > 0$ in the region of determination of a simple wave. In addition, the characteristic equation (1.4) is also the integral of system (1.8)–(1.10). We prove the existence theorem of the solution of problem (1.8)–(1.10). We primarily obtain *a priori* estimates of the solution.

Lemma. *Assume that u, H, and k is the solution of problem (1.8)-(1.10), and the inequalities* $0 < \omega_1 \leqslant \omega = u_{0\lambda} H_0^{-1} \leqslant \omega_2 < \infty$, *where* $\omega_1 = \min_{0\leqslant\lambda\leqslant1} \omega(\lambda)$ *and* $\omega_2 = \max_{0\leqslant\lambda\leqslant1} \omega(\lambda)$, *are satisfied. The estimates*

$$\frac{\sqrt{\omega_1^2\eta^2 + 4\omega_1^2\eta/(\rho\omega_2^2)} - \omega_1\eta}{2} < |u-k| < \frac{\sqrt{\omega_2^2\eta^2 + 4\omega_2\eta/(\rho\omega_1)} + \omega_2\eta}{2} \tag{2.2}$$

are then valid.

Proof. By virtue of (2.1), we have

$$\omega_1\eta \leqslant u_2 - u_1 = \int_0^1 \omega H(\eta,\lambda)\,d\lambda \leqslant \omega_2\eta, \tag{2.3}$$

where $u_2 = u(\eta, 1)$ and $u_1 = u(\eta, 0)$. It follows from (1.4) that

$$\rho(p_0+\eta) > \omega_2^{-1} \int_0^1 u_\lambda (u-k)^{-2} d\lambda = \frac{u_2 - u_1}{\omega_2(u_2 - k)(u_1 - k)}. \tag{2.4}$$

Since $(u_2 - k)(u_1 - k) > 0$ and $u_1 - k = (u_1 - u_2) + (u_2 - k)$, from (2.3) and (2.4) follows the lower estimate

$$|u_2 - k| > \frac{\sqrt{(u_2-u_1)^2 + 4(u_2-u_1)(\rho\omega_2)^{-1}} - (u_2 - u_1)}{2} > \frac{\sqrt{\omega_1^2\eta^2 + 4\omega_1^2\eta\rho^{-1}\omega_2^{-2}} - \omega_1\eta}{2}.$$

Similarly, we obtain the upper estimates: $|u_1 - k| < \left(\sqrt{\omega_2^2\eta^2 + 4\,\omega_2\eta(\rho\omega_1)^{-1}} + \omega_2\eta\right)/2$. The assertion of the lemma follows from the inequalities $|u_2 - k| \leqslant |u-k| \leqslant |u_1 - k|$. The lemma is proved.

We use the existence theorem of a solution of the Cauchy problem for a nonlinear equation in the Banach space B:

$$\frac{dx}{dt} = f(x,t), \qquad x(t_0) = x_0. \tag{2.5}$$

Here $f(x,t)$ is the function of real t and variable $x \in B$ which takes values in the space B.

Let the function $f(x,t)$ be continuous in t and be subject to the conditions $||f(x,t)|| \leqslant M_1$ and $||f(x_1,t) - f(x_2,t)|| \leqslant M_2||x_1 - x_2||$ for $t \in [a,b]$ and $||x - x_0|| \leqslant \theta$. According to [7], there exists a number $[\delta_1 > 0\ \delta_1 = \min(\theta M_1^{-1}, M_2^{-1})]$ such that the Cauchy problem (2.5) has, on the interval $(t_0 - \delta_1, t_0 + \delta_1) \cap [a,b]$, a unique solution $x = \varphi(t)$ left in the sphere: $||\varphi(t) - x_0|| \leqslant \theta$.

To take advantage of the result presented, we consider the Banach space B of the vector functions $\mathbf{U} = (u, H, k)$ of real argument $\lambda \in [0,1]$

$$B = \{(u,H,k)/u \in C^1[0,1], H \in C\,[0,1], k \in R\}$$

with the norm

$$||\mathbf{U}|| = \max_{0\leqslant\lambda\leqslant1} |u_\lambda| + \max_{0\leqslant\lambda\leqslant1} |u| + \max_{0\leqslant\lambda\leqslant1} |H| + |k|,$$

where $C^1[0,1]$ is the set of continuously differentiated functions on the closed interval $[0,1]$, $C\,[0,1]$ is the set of continuous functions, and R is the numerical straight line.

Let $\mathbf{U}_0 = (u_0, H_0, k_0) \in B$. Since u_0 and H_0 are continuous on the closed interval $[0,1]$ and $u_0 - k_0 < 0$ and $H_0 > 0$, there exists a constant $\theta > 0$ such that $|u_0 - k_0| \geqslant |u_0(1) - k_0| > \theta$ and $\min_{0\leqslant\lambda\leqslant1} H_0 > \theta$. In space B, we consider a sphere: $||\mathbf{U} - \mathbf{U}_0|| < \theta/2$. We show that, for $\mathbf{U}$ from the sphere, the inequalities $|u-k| > \theta/2$ and $|H| > \theta/2$ are satisfied. Indeed,

$$|u-k| \geqslant |u_0 - k_0| - ||\mathbf{U} - \mathbf{U}_0|| > \theta/2, \quad \min_{0\leqslant\lambda\leqslant1} |H| > \min_{0\leqslant\lambda\leqslant1} |H_0| - ||\mathbf{U} - \mathbf{U}_0|| > \theta/2. \tag{2.6}$$

By virtue of the continuity of the operator $f(\mathbf{U}, h)$, in the domain (2.6), there exist constants $M_1(\theta, \mathbf{U}_0)$ and $M_2(\theta, \mathbf{U}_0)$ such that

$$||f(\mathbf{U},\eta)|| \leqslant M_1(\theta,\mathbf{U}_0), \quad ||f(\mathbf{U}_2,\eta) - f(\mathbf{U}_1,\eta)|| \leqslant M_2(\theta,\mathbf{U}_0)||\mathbf{U}_2 - \mathbf{U}_1||. \tag{2.7}$$

Using the above result, we establish the fact of the existence and uniqueness of the solution of problem (1.8)–(1.10) on the interval $[\eta_0 - \delta_1, \eta_0 + \delta_1]$ in space B.

Theorem. *Let u_0 and H_0 satisfy the conditions of the lemma. The solution of problem* (1.8)–(1.10) *exists on any finite interval $\eta \in [\delta, A]$, where $\delta > 0$ and $A < \infty$, and belongs to the space B.*

Proof. With $\mathbf{U} = (u, H, k) \in B$ and $\eta \in [\delta, A]$, by virtue of (2.2), the inequalities

$$\frac{\sqrt{\omega_1^2\delta^2 + 4\,\omega_1^2\delta/(\rho(p_0 + A)\,\omega_1^2)} - \omega_1\delta}{2} < |u - k| < \frac{\sqrt{\omega_2^2A^2 + 4\,\omega_2A/(\rho(p_0 + \delta)\,\omega_1)} + \omega_2A}{2} \tag{2.8}$$

hold true. Having differentiated the first equation in (1.8) with respect to λ and having integrated over η, we obtain

$$u_\lambda(\eta, \lambda) = u_{0\lambda}(\lambda) \exp\Big\{\int_{\eta_0}^{\eta} \rho^{-1}(s)(u - k)^{-2} ds\Big\}. \tag{2.9}$$

After integration of the second equation in (1.8), we find

$$H(\eta, \lambda) = H_0(\lambda)\, \exp\Big\{\int_{\eta_0}^{\eta} \rho^{-1}(s)(u - k)^{-2} ds\Big\}. \tag{2.10}$$

After that, by virtue of (2.8)–(2.10) the conditions (2.7) are satisfied with the same constants M_1 and M_2, which are dependent only on δ, A, and $\mathbf{U}_0 = (u_0, H_0, k_0)$. Therefore, having constructed the solution on the interval $[\eta_0 - \delta_1, \eta_0 + \delta_1]$, one can extend it uniquely over the entire interval $[\delta, A]$. The theorem is proved.

The simple wave will be constructed if one solves the equation

$$\eta_t + \dot{k}(\eta)\,\eta_x = 0. \tag{2.11}$$

According to the known facts of the theory of quasi-linear equations [8], the properties of the solutions (2.11) depend on whether the derivative $k'(\eta)$ is of a fixed sign [Eq. (2.11) satisfies the convexity condition or does not satisfy it]. It follows from Eq. (2.11) that along the characteristics $dx/dt = k$

$$\eta_x = \eta_{0x}(1 + tk'(\eta_0)\,\eta_{0x}(x))^{-1}. \tag{2.12}$$

It follows from (2.12) that if $k'(\eta_0)\,\eta_0'(x) > 0$, the derivative η_x remains bounded ($|\eta_x| \leqslant |\eta_{0x}|$), and the solution of Eq. (2.11) exists for any t. If $k'(\eta_0)\,\eta_0'(x) < 0$, the solution of (2.11) exists only for finite t. Let us consider the sign of the function $k'(\eta)$. If the function of the equation of state $\rho = \rho(p_0 + \eta)$ is such that the inequality,

$$3\,\rho - \eta\rho'(p_0 + \eta) > 0, \tag{2.13}$$

which is an analog of the normality condition of the gas for system (1.2) [6], is satisfied, then $k'(\eta) > 0$. Indeed, since

$$\rho^2 = \Big(\int_0^1 H(u - k)^{-2} d\nu\Big)^2 \leqslant \eta \int_0^1 H(u - k)^{-4} d\nu,$$

the numerator on the right-hand side of the equation for k (1.9) is negative:

$$\rho' - \frac{3}{\rho}\int_0^1 H(u - k)^{-4} d\nu \leqslant \rho' - \frac{3\rho}{\eta} < 0.$$

Since $u - k < 0$, we obtain $k'(\eta) > 0$. In the general case, as is shown in [6], the inequality (2.13) can be violated even when the initial equation of state of a barotropic medium is subject to the convexity condition. In a definite range of η variation, the derivative $k'(\eta)$ can change sign. In the neighborhood of the points of sign alternation, $k(\eta)$ has either a maximum or a minimum [$k''(\eta) \neq 0$ is assumed at these points].

Let the condition (2.13) be not satisfied. From the inequality

$$\rho' - \frac{3}{\omega_1 \rho} \int_0^1 \frac{u_\nu}{(u-k)^4}\, d\nu \leqslant \rho' - \frac{3}{\rho} \int_0^1 \frac{H}{(u-k)^4}\, d\nu \leqslant \rho' - \frac{3}{\omega_2 \rho} \int_0^1 \frac{u_\nu}{(u-k)^4}\, d\nu,$$

with allowance for the *a priori* estimates of the lemma $(u_2-u_1)(\rho\omega_2)^{-1} \leqslant (u_1-k)(u_2-k) \leqslant (u_2-u_1)(\rho\omega_1)^{-1}$, we obtain the following new inequalities:

$$\rho' - \frac{3\,\omega_2^2}{\rho\omega_1^2} - \rho^2 \frac{\omega_2^3}{\omega_1} \leqslant \rho' - \frac{3}{\rho} \int_0^1 \frac{H d\nu}{(u-k)^4} \leqslant \rho' - \frac{3\,\omega_1^2}{\rho\omega_2^2} - \rho^2 \frac{\omega_1^3}{\omega_2}.$$

If the function $\rho = \rho(p_0 + \eta)$ and the initial data are such that the inequality

$$\rho' - 3\,\omega_1^2/\rho\omega_2^2 - \rho^2\omega_1^3/\omega_2 < 0 \tag{2.14}$$

is satisfied, then $k'(\eta) > 0$ is in the domain of determination of a simple wave, despite the fact that (2.13) is not satisfied. If the inequality

$$\rho' - 3\omega_2^2/\rho\omega_1^2 - \rho^2\omega_2^3/\omega_1 > 0$$

is satisfied on a part of the interval of determination of the simple wave, and the inequality (2.14) is satisfied on the other part, $k'(\eta)$, undoubtedly, changes sign in the simple-wave region.

Let us call the simple wave a *wave decreasing (increasing) the fluid level* if the inequality $\eta_t + u(\eta,1)\,\eta_x = (u(\eta,1)-k)\,\eta_x < 0$ (> 0) is satisfied and call it a *centered wave* if $k(\eta) = x/t$.

It follows from (2.12) that if $k'(\eta) > 0$, the gradient catastrophe will occur in the wave increasing the fluid level, and the waves decreasing the fluid level exist for any $t > 0$. If $k'(\eta) < 0$, the waves increasing the fluid level exist for any t, and those decreasing the fluid level exist only for finite t. The centered waves for $t > 0$ are the waves decreasing the fluid level ($\eta < \eta_0$) if $k'(\eta) > 0$ for any η and the waves increasing the fluid level ($\eta > \eta_0$) if $k'(\eta) < 0$. If $k'(\eta)$ changes sign, the simple centered wave will be defined only for $\eta \in [\eta_0, \eta_*]$, where $k'(\eta_*) = 0$ (in the centered wave, the slope of the characteristics will change monotonically).

Thus, the wave exists either for any $t > 0$ or only for finite t, depending on the initial equation of state and on the initial data. Sign alternation of the derivative $k'(h)$ has an analog in gas dynamics with anomalous thermodynamic properties. In the general case, simple waves decreasing and increasing the fluid level which are determined for all $t > 0$ can exist.

3. Exact Solutions. Let us consider the problem of construction of a simple wave (1.18) and (1.19) for a medium with a polytropic equation of state: $\rho = p^{2\gamma}$ $(0 < \gamma < 1/2)$. In this case, relations (1.15) and (1.16) are integrated as follows:

$$h = (2(\beta-1)g)^{-1}\eta^{2(\beta-1)}, \qquad Y/h = 1 - (p/\eta)^{2(\beta-1)}. \tag{3.1}$$

Equation (1.18) admits a one-parameter group of extensions: $p \to bp$, $\eta \to b\eta$, and $\Psi \to b^{3/2-\gamma}\Psi$ (b is the parameter of the group). We construct the solution of Eq. (1.18) that is invariant under the indicated group. According to the known algorithm of searching for invariant solutions, we assume that

$$\Psi(\eta, p) = \eta^{3/2-\gamma}\Psi_1(p/\eta), \tag{3.2}$$

where Ψ_1 is a new unknown function of the argument. Let $\nu = p/\eta$, $\beta = 3/2 - \gamma$, and $A = \Psi_1(1)$. Substituting (3.2) into (1.18), we obtain the following equation for determination of Ψ_1:

$$\beta(\nu A - \Psi_1)\,\Psi_1'' + (\beta-1)(\Psi_1')^2 - \beta(\beta-1)\,A\Psi_1' + 1 = 0. \tag{3.3}$$

The boundary condition (1.19) is transformed to the form $\Psi_1(0) = 0$. Owing to (3.3), the function $\Phi(\nu) = \nu A - \Psi_1(\nu)$ should satisfy the equation

$$-\beta\Phi\Phi'' + (\beta-1)(A-\Phi')^2 - \beta(\beta-1)\,A^2 + \beta(\beta-1)\,\Phi' + 1 = 0 \tag{3.4}$$

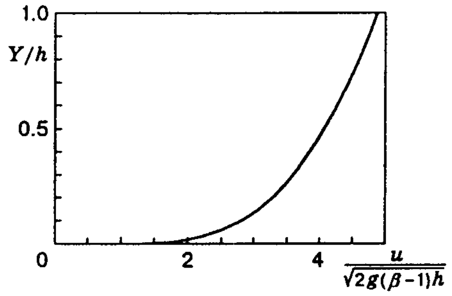

Fig. 1

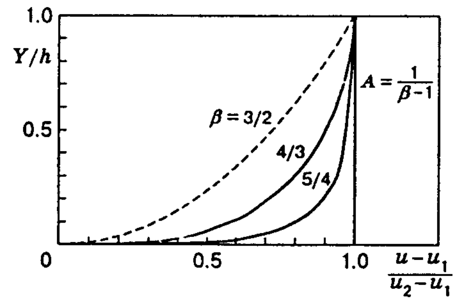

Fig. 2

and be subject to the boundary conditions $\Phi(0) = \Phi(1) = 0$. After the substitution $\Phi' = L(\Phi)$, Eq. (3.4) is integrated as follows:

$$\Phi = C(b_1 - L)^{\alpha}(L - b_2)^{\delta}. \tag{3.5}$$

Here b_1 and b_2 are the roots of the square equation, $L^2 + (\beta - 2)AL + (1 - \beta)A^2 + 1/(\beta - 1) = 0$, $\delta = -\beta b_2(\beta - 1)^{-1}(b_1 - b_2)^{-1}$, $\alpha = \beta b_1(\beta - 1)^{-1}(b_1 - b_2)^{-1}$, and C is an arbitrary constant. The boundary conditions for the function Φ are satisfied at the points $L = b_1$ and $L = b_2$ under the condition that $\alpha > 0$ and $\delta > 0$. Let $\nu = 0$ and 1 correspond to $L = b_2$ and b_1. Since $L = d\Phi/d\nu$, we have from (3.5)

$$\nu = \frac{p}{\eta} = \int\limits_0^{(L-b_2)(b_1-b_2)^{-1}} z^{\delta-1}(1-z)^{\alpha-1}dz \Big/ \int\limits_0^1 z^{\delta-1}(1-z)^{\alpha-1}\,dz. \tag{3.6}$$

With allowance for the above replacements, we find the horizontal velocity u and the velocity of the characteristics k:

$$u = \Psi_p = (\eta^{\beta}(\nu A - \Phi))'_p = \eta^{\beta-1}(A - L), \qquad k = \beta A\eta^{\beta-1}. \tag{3.7}$$

By virtue of (3.1) and (3.7), from (3.6) we obtain the following relationship between u and Y:

$$\frac{Y}{h} = 1 - \left(\int\limits_0^{(A-b_2-u/\sqrt{2g(\beta-1)h})(b_1-b_2)^{-1}} z^{\delta-1}(1-z)^{\alpha-1}\,dz \Big/ \int\limits_0^1 z^{\delta-1}(1-z)^{\alpha-1}\,dz\right)^{2(\beta-1)}. \tag{3.8}$$

Formula (3.8) determines the velocity profile in the simple wave $u = u(h(x,t), Y)$. The velocity varies from $u = u_1 = \sqrt{2g(\beta-1)h}(A - b_1)$ at the bottom $[Y = 0\ (p = \eta(x,t))]$ to $u = u_2 = \sqrt{2g(\beta-1)h}(A - b_2)$ on a free surface $[Y = h(X,T)\ (p = 0)]$. For satisfaction of the inequality $\alpha > 0$ and $\delta > 0$, it suffices to require that $A^2 > (\beta - 1)^{-2}$. If $A > (\beta - 1)^{-1} > 0$, then $k > u_{\max} = u_2$. If $A < -(\beta - 1)^{-1} < 0$, then $k < u_{\min} = u_1$. The case $A^2 = (\beta - 1)^{-2}$ yields $b_2 = 0$, $u = \eta^{\beta-1}A$, and $k = \beta A\eta^{\beta-1}$, which corresponds to a shear-free flow ($u_Y \equiv 0$). Note that $\beta = 3/2$ corresponds to an incompressible fluid $[\rho(p) = \text{const}]$. In this case, solution (3.8) coincides, up to insignificant transformations, with Freeman's solution [2]. Here the parameter $(2b_1 + b_2)/(4b_2 + 2b_1)$ corresponds to the parameter a^2 from [2].

Figure 1 shows the profile of the horizontal velocity for $\beta = 4/3$, $A^2 = 17$, and $A > 0$. Formula (3.8) determines Y/h as a function of $(u - u_1)/(u_2 - u_1)$.

Figure 2 shows the diagrams of the dependence of $(u - u_1)/(u_2 - u_1)$ on Y/h for $\beta = 3/2$, $4/3$, and $5/4$ and $A = 5$. The vertical straight line corresponds to $A = (\beta - 1)^{-1}$ (shear-free flow). In the case of a centered simple wave ($k(\eta) = x/t$)), from (3.1) and (3.7) we obtain the shape of the free surface: $h(X,T) = (2\beta^2(\beta - 1)\,A^2 g)^{-1}(X/T)^2$.

Figure 3 shows the diagrams of the free surface for $\beta = 4/3$ and $A^2 = 17$ and of the contact surfaces $[Y = Y(\mu, X/T)\ (\mu = \text{const})]$ for various μ (the value of μ is determined as follows: $\mu = Y/h_0$ for $X = k_0T$).

As a result, we have found that a simple wave that is uniquely defined by a given pressure distribution

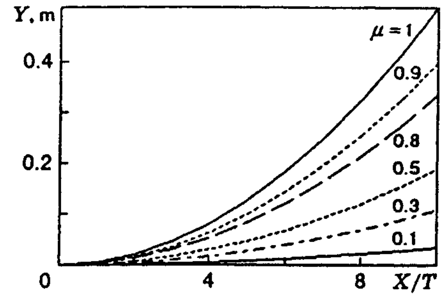

Fig. 3

at the bottom for $t = 0$ can be adjacent on the characteristic to any shear flow with a monotone-in-depth velocity profile. The simple wave has been found either to exist for any $t > 0$ or decay, depending on the properties of the monotone functions $k(\eta)$ and $\eta(x, 0)$. In a barotropic medium, with the equation of state, which is subject to the convexity condition, one can observe the anomalous behavior of simple waves: there are waves increasing the fluid level that do not decay for $t > 0$. We have obtained a class of exact solutions of the system of simple-wave equations that describe simple waves propagating in a barotropic fluid layer with velocity $k = \beta A\sqrt{2(\beta - 1)\,gh}$. This solution is a generalization of the simple waves in an incompressible fluid that were found by Freeman [2].

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